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Particular Solutions for Linear Third Order Differential Equations by N-Fractional Calculus Operator N^{γ}

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Abstract

There are many articles that are concerned with the application of fractional calculus for solving differential equations. In this paper, we apply the N-fractional calculus operator N^{γ} developed by Nishimoto to solve some liner third order differential equations.

Key words: Fractional calculus, homogeneous, particular solution, operator N^{γ}

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I. Definitions and Lemmas [1]

There are many definitions for the fractional integral of a function of one or many variables. In this paper we will use the definition given by K. Nishimoto, which is based on the Goursat's theorem (Cauchy's formula) :

$$f^{(n)}(z)=\frac{n!}{2\pi i}\int_{C}\frac{f(\varsigma)}{(\varsigma-z)^{n+1}}d\varsigma \qquad n\in N\cup\{0\}, z\in D$$

where C and D are defined below.

1. Definition of fractional derivative

If $f(z)$ is an analytic (regular) function and it has no branch point inside C, and on C, $(C=\{C-,C+\})$, then

$$_{C-}f_{\nu}={}_{C-}f_{\nu}(z)=\frac{\Gamma(\nu+1)}{2\pi i}\int_{C-}\frac{f(\varsigma)}{(\varsigma-z)^{\nu+1}}d\varsigma$$

$$=\frac{\Gamma(\nu+1)}{2\pi i}\int_{-\infty}^{(0+)}\eta^{-(\nu+1)}f(z+\eta)d\eta, \qquad (\varsigma-z=\eta)$$

$$(\varsigma\neq z,\ -\pi\leq\arg(\varsigma-z)\leq\pi,\ \nu\neq Z^{-})$$

$$_{C+}f_{\nu}={}_{C+}f_{\nu}(z)=\frac{\Gamma(\nu+1)}{2\pi i}\int_{C+}\frac{f(\varsigma)}{(\varsigma-z)^{\nu+1}}d\varsigma$$

$$=\frac{\Gamma(\nu+1)}{2\pi i}\int_{\infty}^{(0+)}\eta^{-(\nu+1)}f(z+\eta)d\eta, \qquad (\varsigma-z=\eta)$$

$$(\varsigma\neq z,\ 0\leq\arg(\varsigma-z)\leq 2\pi,\ \nu\neq Z^{-})$$

$$f_{-n}={}_{C}f_{-n}=\lim_{\nu\to -n}{}_{C}f_{\nu} \qquad n\in Z^{+}, C=\{C-,C+\}$$

where $C-$ and $C+$ are integral curves. $C-$ is a curve along the cut joining two points z and $-\infty+i\,\mathrm{Im}(z)$, and $C+$ is a curve along the cut joining two points z and $\infty+i\,\mathrm{Im}(z)$. Then

$$f_{\nu}={}_{C}f_{\nu}(z)=\{{}_{C-}f_{\nu},\ {}_{C+}f_{\nu}\} \qquad (\nu>0)$$

is the fractional derivative of order ν of the function $f(z)$. $(\nu\in R \textit{ and } z\in C)$

2. Definition of fractional integral

$f_{\nu}(\nu<0)$ is the fractional integral of order $|\nu|$. In other words, the derivative of fractional order $-\nu(\nu>0)$ is the fractional integral of order $\nu(\nu\in R)$.

3. The set $\mathcal{F}$

We call the function $f = f(z)$ such that $|f_\nu|<\infty$ in D as the fractional differintegrable function by arbitrary order ν and denote the set of them with notation $\mathcal{F}$. That is,

$$|f_\nu|<\infty \Leftrightarrow f\in\mathcal{F} \quad (\text{in } D)$$

where $D=\{D-,D+\}$, $D-$ is a domain surrounded by $C-$ and $D+$ is a domain surrounded by $C+$, both contain the points over the curve C.

4. The operator N^{γ} [2]

By operating a function (UV) by N^{γ}, we mean

$$N^{\gamma}(UV)=(UV)_\gamma=\sum_{k=1}\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-k)\Gamma(k+1)}U_{\gamma-k}V_k$$

The last equality is based on lemma 3 below.

The following lemmas are needed in the subsequent discussions.

Lemma 1. (Linearity) Let $U(z)$ and $V(z)$ be analytic and one valued function. If U_ν, V_ν exist, then

(i) $(aU+bU)_\nu=aU_\nu+bU_\nu$ (a, b are constants, $z\in C, \nu\in R$)

(ii) $(aU)_\nu=aU_\nu$

Lemma 2. (Index law) Let $f(z)$ be an analytic and single valued function. If $(f_\mu)_\nu$, $(f_\nu)_\mu$ exist, then

$$(f_\mu)_\nu=(f_\nu)_\mu=f_{\mu+\nu} \qquad f_\nu,\ f_\mu\neq 0$$

Lemma 3. Let $U(z)$ and $V(z)$ be analytic and one valued functions. If U_ν and V_ν exist, then

$$(UV)_\nu=\sum_{k=1}\frac{\Gamma(\nu+1)}{\Gamma(\nu+1-k)\Gamma(k+1)}U_{\nu-k}V_k$$

where $\nu\in R$ and $$\left|\frac{\Gamma(\nu+1)}{\Gamma(\nu+1-k)\Gamma(k+1)}\right|<\infty$$

Lemma 4. $(e^{az})_\nu=a^\nu e^{az},\ a\neq 0,\ z\in C,\ \nu\in R$

II. The forms of Particular solutions for linear third order non-homogeneous differential equations

Theorem 1. If $f \in \mathcal{F}$ and $f_{-\gamma-1} \neq 0$, then the linear non-homogeneous third order equation

$$\Psi_3(z-a)(z-b)(z-c)+\Psi_2[Dz^2+Ez+G]+\Psi_1[Hz+J]+\gamma(\gamma-1)(D-2\gamma-2)\Psi = f \tag{A}$$

where *a, b, c, D, E, G, H* and *J* are given constants and satisfy the condition (A-11) has a particular solution of the form

1. $$\Psi = \left\{(z-a)^{-R}(z-b)^{-S}(z-c)^{-T}\left[(z-a)^{R-1}(z-b)^{S-1}(z-c)^{T-1}f_{-\gamma}\right]_{-1}\right\}_{\gamma-2}$$

for $a \neq b \neq c$; (A-1)

2. $$\Psi = \left\{(z-a)^{-L}(z-c)^{-N}\exp\left(\frac{M}{z-a}\right)\left[(z-a)^{L-2}(z-c)^{N-1}\exp\left(-\frac{M}{z-a}\right)f_{-\gamma}\right]_{-1}\right\}_{\gamma-2}$$

for $a = b \neq c$; and (A-2)

3. $$\Psi = \left\{(z-a)^{-X}\exp\left(\frac{2(z-a)Y+Z}{2(z-a)^2}\right)\left[(z-a)^{X-3}\exp\left(-\frac{2(z-a)Y+Z}{2(z-a)^2}\right)f_{-\gamma}\right]_{-1}\right\}_{\gamma-2}$$

for $a = b = c$, (A-3)

where
$$\gamma = \frac{3J-AH}{3E-2AD} \tag{A-4}$$

$$A = -(a+b+c),\ B = ab+bc+ca,\ C = -abc \tag{A-5}$$

$$R = \frac{(D-3\gamma)a^2+(E-2A\gamma)a+G-B\gamma}{(a-b)(a-c)} \tag{A-6}$$

$$S = \frac{(D-3\gamma)b^2+(E-2A\gamma)b+G-B\gamma}{(b-c)(b-a)} \tag{A-7}$$

$$T = \frac{(D-3\gamma)c^2+(E-2a\gamma)c+G-B\gamma}{(c-a)(c-b)} \tag{A-8}$$

$$L = (D-3\gamma)-N,\ M = R(a-b),\ N = T\cdot\frac{c-b}{c-a} \tag{A-9}$$

$$X = (D-3\gamma),\ Y = 2(D-3\gamma)a+(E-2A\gamma),\ Z = R(a-b)(a-c) \tag{A-10}$$

$$(3J-AH)^2 = (3E-2AD)[(A-E)H-(3-2D)J] \tag{A-11}$$

Proof: Suppose the non-homogeneous third order differential equation is

$$\Psi_3(z^3+Az^2+Bz+C)+\Psi_2[Dz^2+Ez+G]+\Psi_1[Hz+J]+K\Psi=f \tag{1}$$

where
$$z^3+Az^2+Bz+C=(z-a)(z-b)(z-c)\neq 0$$
$$A=-(a+b+c),\ B=ab+bc+ca,\ C=-abc. \tag{1a}$$
and
$$K=\gamma(\gamma-1)(D-2\gamma-2) \tag{1b}$$
Let $\Psi=\omega_{\gamma}$, then $\Psi_1=\omega_{\gamma+1}$, $\Psi_2=\omega_{\gamma+2}$, $\Psi_3=\omega_{\gamma+3}$ [3-6] and (1) becomes

$$\omega_{\gamma+3}(z^3+Az^2+Bz+C)+\omega_{\gamma+2}(Dz^2+Ez+G)+\omega_{\gamma+1}(Hz+J)+K\omega_{\gamma}=f \tag{2}$$

By applying the operator N^{γ} [2,7] to $\omega_3(z^3+Az^2+Bz+C)$, we have

$$N^{\gamma}[\omega_3(z^3+Az^2+Bz+C)]=[\omega_3(z^3+Az^2+Bz+C)]_{\gamma}$$
$$=\sum_{k=0}^{3}\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-k)\Gamma(k+1)}\omega_{\gamma+3-k}(z^3+Az^2+Bz+C)_k$$
$$=\omega_{\gamma+3}(z^3+Az^2+Bz+C)+\gamma\omega_{\gamma+2}(3z^2+2Az+B)$$
$$+\frac{\gamma(\gamma-1)}{2!}\omega_{\gamma+1}(6z+2A)+\frac{\gamma(\gamma-1)(\gamma-2)}{3!}\omega_{\gamma}\cdot 6 \tag{3}$$

Substitute this into (2), we have

$$[\omega_3(z^3+Az^2+Bz+C)]_{\gamma}+\omega_{\gamma+2}[(Dz^2+Ez+G)-\gamma(3z^2+2Az+B)]$$
$$+\omega_{\gamma+1}[(Hz+J)-\gamma(\gamma-1)(3z+A)]+[K-\gamma(\gamma-1)(\gamma-2)]\omega_{\gamma}=f \tag{4}$$

By applying the operator N^{γ} to $\omega_2[(Dz^2+Ez+G)-\gamma(3z^2+2Az+B)]$, we have

$$\left\{\omega_2\,[(Dz^2+Ez+G)-\gamma(3z^2+2Az+B)]\right\}_{\gamma}$$
$$=\sum_{k=0}^{2}\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-k)\Gamma(k+1)}\omega_{\gamma+2-k}\left[(Dz^2+Ez+G)-\gamma(3z^2+2Az+B)\right]_k$$
$$=\omega_{\gamma+2}\left|(Dz^2+Ez+G)-\gamma(3z^2+2Az+B)\right|$$
$$+\gamma\omega_{\gamma+1}[2Dz+E-\gamma(6z+2A)]+\frac{\gamma(\gamma-1)}{2!}\omega_{\gamma}(2D-6\gamma) \tag{5}$$

Substitute this into (4), we obtain

$$\left\{\omega_3(z^3+Az^2+Bz+C)+\omega_2[(Dz^2+Ez+G)-\gamma(3z^2+2Az+B)]\right\}_\gamma$$
$$+\omega_{\gamma+1}[(Hz+J)-\gamma(\gamma-1)(3z+A)-\gamma(2Dz+E-\gamma(6z+2A))]$$
$$+[K-\gamma(\gamma-1)(\gamma-2)-\gamma(\gamma-1)(D-3\gamma)]\omega_\gamma=f \tag{6}$$

Substitute (1b) into (6), we see that the coefficient of ω_γ equals to zero. By setting the coefficients of $\omega_{\gamma+1}$ to zero, we have

$$\left\{\omega_3(z^3+Az^2+Bz+C)+\omega_2[(D-3\gamma)z^2+(E-2A\gamma)z+G-B\gamma]\right\}_\gamma=f \tag{7}$$

Let $U=\omega_2$, then $U_1=\omega_3$, and (7) becomes

$$U_1+\frac{(D-3\gamma)z^2+(E-2A\gamma)z+G-B\gamma}{(z-a)(z-b)(z-c)}U=\frac{f_{-\gamma}}{(z-a)(z-b)(z-c)}, \tag{8}$$

1. In case $a\neq b\neq c$, let

$$p(z)=\frac{(D-3\gamma)z^2+(E-2A\gamma)z+G-B\gamma}{(z-a)(z-b)(z-c)}=\frac{R}{(z-a)}+\frac{S}{(z-b)}+\frac{T}{(z-c)} \tag{9}$$

where R, S and T are given by (A-6) (A-7) (A-8).

Then
$$\int p(z)dz=\ln[(z-a)^R(z-b)^S(z-c)^T]$$

and
$$U=\exp(-\int pdz)\left[\exp(\int pdz)\frac{f_{-\gamma}}{(z-a)(z-b)(z-c)}\right]_{-1}$$
$$=(z-a)^{-R}(z-b)^{-S}(z-c)^{-T}\left[(z-a)^{R-1}(z-b)^{S-1}(z-c)^{T-1}f_{-\gamma}\right]_{-1} \tag{10}$$

Hence we obtain a particular solution for (A) of the form

$$\Psi=U_{\gamma-2}=\left\{(z-a)^{-R}(z-b)^{-S}(z-c)^{-T}\left[(z-a)^{R-1}(z-b)^{S-1}(z-c)^{T-1}f_{-\gamma}\right]_{-1}\right\}_{\gamma-2}$$

2. In case $a=b\neq c$, let

$$p(z)=\frac{(D-3\gamma)z^2+(E-2A\gamma)z+G-B\gamma}{(z-a)(z-b)(z-c)}=\frac{L}{(z-a)}+\frac{M}{(z-a)^2}+\frac{N}{(z-c)} \tag{11}$$

where L, M, and N are given by (A-9)

Then
$$\int p(z)dz=\ln\left[(z-a)^L(z-c)^N\exp(-\frac{M}{z-a})\right]$$

and $$U=\exp(-\int pdz)\left[\exp(\int pdz)\frac{f_{-\gamma}}{(z-a)^2(z-c)}\right]_{-1}$$

$$=(z-a)^{-L}(z-c)^{-N}\exp(\frac{M}{z-a})\left[(z-a)^{L-2}(z-c)^{N-1}\exp(-\frac{M}{z-a})f_{-\gamma}\right]_{-1} \quad (12)$$

Hence we obtain a particular solution for (A) of the form

$$\Psi=\left\{(z-a)^{-L}(z-c)^{-N}\exp(\frac{M}{z-a})\left[(z-a)^{L-2}(z-c)^{N-1}\exp(\frac{-M}{z-a})f_{-\gamma}\right]_{-1}\right\}_{\gamma-2}$$

3. In case $a=b=c$, let

$$p(z)=\frac{(D-3\gamma)z^2+(E-2A\gamma)z+G-B\gamma}{(z-a)^3}=\frac{X}{(z-a)}+\frac{Y}{(z-a)^2}+\frac{Z}{(z-a)^3} \quad (13)$$

where X, Y and Z are given by (A-10).

Then $$\int p(z)dz=\ln\left[(z-a)^X\exp(-\frac{2(z-a)Y+Z}{2(z-a)^2})\right]$$

and $$U=(z-a)^{-X}\exp(\frac{2(z-a)Y+Z}{2(z-a)^2})\left[(z-a)^{X-3}\exp(-\frac{2(z-a)Y+Z}{2(z-a)^2})f_{-\gamma}\right]_{-1} \quad (14)$$

Hence we obtain a particular solution for (A) of the form

$$\Psi=\left\{(z-a)^{-X}\exp(\frac{2(z-a)Y+Z}{2(z-a)^2})\left[(z-a)^{X-3}\exp(-\frac{2(z-a)Y+Z}{2(z-a)^2})f_{-\gamma}\right]_{-1}\right\}_{\gamma-2}$$

To find γ, recall that the $\omega_{\gamma+1}$ term is set to zero. From (6),

$$[H-3\gamma(\gamma-1)-2\gamma D+6\gamma^2]z+[J-\gamma(\gamma-1)A-\gamma E+\gamma^2 2A]=0 \quad (15)$$

This may be simplified as

$$3\gamma^2+(3-2D)\gamma+H=0 \quad (16)$$

$$A\gamma^2+(A-E)\gamma+J=0 \quad (17)$$

In order to find γ that satisfies (16) and (17), we try to find the common factor of two equations

$A(16)-3(17)$: $$(3E-2AD)\gamma+(AH-3J)=0 \quad (18)$$

$J(16)-H(17)$: $$(3J-AH)\gamma+[(3-2D)J-(A-E)H]=0 \quad (19)$$

Since the coefficients of the two equations must be proportional, that is,

$$\frac{3E-2AD}{3J-AH}=\frac{AH-3J}{(3-2D)J-(A-E)H} \tag{20}$$

Hence we have $(3J-AH)^2=(3E-2AD)[(A-E)H-(3-2D)J]$, (A-11)

and by (18),
$$\gamma=\frac{3J-AH}{3E-2AD} \tag{21}$$

This completes the proof.

If the condition (A-11) is not satisfied, we try to solve γ from (16)

$$\gamma=\frac{-(3-2D)\pm\sqrt{(3-2D)^2-12H}}{6} \tag{22}$$

Then we choose J by (17): $J=-A\gamma^2-(A-E)\gamma$ (23)

Hence we have the following

Theorem 2. If $f\in\mathcal{F}$ and $f_{-\gamma-1}\neq 0$, then the non-homogeneous equation (A) has a particular solution of the form (A-1) to (A-3), provided J satisfies the conditions (23), where a, b, c, D, E, G and H are arbitrary given constants, and γ is given by (22). $R, S,\ldots, Z$ are given by (A-6) to (A-10).

The γ could also be solved from (17):

$$\gamma=\frac{-(A-E)\pm\sqrt{(A-E)^2-4AJ}}{2A},\quad A\neq 0. \tag{24}$$

Then H is restricted by (16): $H=-3\gamma^2-(3-2D)\gamma$. (25)

Hence we have the following

corollary 2-1. The equation (A) has a particular solution as above, provided H satisfies the conditions (25), where γ is given by (24). The remaining parameters (coefficients in (A)) are arbitrary.

III. Particular solutions for some special types of third order linear non-homogeneous differential equations.

In order to have a solution for the equation (A) by the method described in the preceding section, it is necessary to set $\omega_{\gamma+1}$ to be zero. We have mentioned two ways to do this. Here is another way. Recall that

$$\gamma=\frac{-(3-2D)\pm\sqrt{(3-2D)^2-12H}}{6}.$$

Choosing the coefficients of (1) such that

$$(3-2D)^2-12H=0 \tag{26}$$

Then
$$\gamma=\frac{1}{3}D-\frac{1}{2} \tag{27}$$

Hence
$$D=3\gamma+\frac{3}{2},\ H=3\gamma^2.$$

From (24),
$$\gamma=\frac{-(A-E)\pm\sqrt{(A-E)^2-4AJ}}{2A},\quad A\neq 0.$$

Choosing the coefficients of (1) such that
$$(A-E)^2-4AJ=0, \tag{28}$$

we have
$$E=(1+2\gamma)A,\ J=A\gamma^2$$

From (1b),
$$K=\gamma(\gamma-1)(D-2\gamma-2),$$

hence
$$K=\gamma(\gamma-1)(\gamma-\frac{1}{2})$$

And the parameters of R to Z become

$$R=\frac{1.5a^2-(a+b+c)a+G-\gamma(ab+bc+ca)}{(a-b)(a-c)} \tag{29-1}$$

$$S=\frac{1.5b^2-(a+b+c)b+G-\gamma(ab+bc+ca)}{(b-c)(b-a)} \tag{29-2}$$

$$T=\frac{1.5c^2-(a+b+c)c+G-\gamma(ab+bc+ca)}{(c-a)(c-b)} \tag{29-3}$$

$$L=1.5-N,\ M=R(a-b),\ N=T\cdot\frac{c-b}{c-a} \tag{30}$$

$$X=1.5,\ Y=2a-b-c,\ Z=R(a-b)(a-c) \tag{31}$$

Theorem 3. If $f\in\mathcal{F}$ and $f_{-\gamma-1}\neq 0$, then the following type of non-homogeneous third order differential equation

$$\Psi_3(z-a)(z-b)(z-c)+\Psi_2[3(\gamma+\frac{1}{2})z^2-(a+b+c)(1+2\gamma)z+G]$$
$$+\gamma^2\Psi_1[3z-(a+b+c)]+\gamma(\gamma-1)(\gamma-\frac{1}{2})\Psi=f \tag{A1}$$

where *a, b, c* and *G* are arbitrary constants, has a particular solution of the form (A-1) to (A-3), where *R* to *Z* are given by (29) to (31).

Since the choice of coefficients of (1) to make the $\omega_{\gamma+1}$ term (in (6)) zero is not restricted by (26) or (28), so we can obtain particular solutions to different types of equations of the form (A) . For instance, in (22), let

$$\sqrt{(3-2D)^2-12H}=3 \tag{32}$$

Then $6\gamma=-3+2D\pm 3$, so we have

(i)

$$\gamma=\frac{1}{3}D \tag{33}$$

and

$$H=3\gamma(\gamma-1), K=\gamma(\gamma-1)(\gamma-2)$$

$$D-3\gamma=0,\ \ E-2A\gamma=A=-(a+b+c)$$

$$R=\frac{-a(a+b+c)+G-(ab+bc+ca)\gamma}{(a-b)(a-c)} \tag{34-1}$$

$$S=\frac{-b(a+b+c)+G-(ab+bc+ca)\gamma}{(b-c)(b-a)} \tag{34-2}$$

$$T=\frac{-c(a+b+c)+G-(ab+bc+ca)\gamma}{(c-a)(c-b)} \tag{34-3}$$

$$L=-N,\ M=R(a-b),\ N=T\cdot\frac{c-b}{c-a} \tag{35}$$

$$X=0,\ Y=-(a+b+c),\ Z=R(a-b)(a-c) \tag{36}$$

Or (ii)

$$\gamma=\frac{1}{3}D-1 \tag{37}$$

$$H=3\gamma(\gamma+1),\ K=\gamma(\gamma-1)(\gamma+1)$$

$$D-3\gamma=3, E-2A\gamma=A=-(a+b+c)$$

$$R=\frac{3a^2-a(a+b+c)+G-(ab+bc+ca)\gamma}{(a-b)(a-c)} \tag{38-1}$$

$$S=\frac{3b^2-b(a+b+c)+G-(ab+bc+ca)\gamma}{(b-c)(b-a)} \tag{38-2}$$

$$T=\frac{3c^2-c(a+b+c)+G-(ab+bc+ca)\gamma}{(c-a)(c-b)} \tag{38-3}$$

$$L=3-N,\ M=R(a-b),\ N=T\cdot\frac{c-b}{c-a} \tag{39}$$

$$X=3,\ Y=5a-b-c,\ Z=R(a-b)(a-c) \tag{40}$$

Hence we have the following

Corollary 3-1. If $f\in\mathcal{F}$ and $f_{-\gamma-1}\neq 0$, then the following of non-homogeneous third order differential equation

$$\Psi_3(z-a)(z-b)(z-c)+\Psi_2[3\gamma z^2-(a+b+c)(1+2\gamma)z+G]$$

$$+\Psi_1[3\gamma(\gamma-1)z-(a+b+c)\gamma^2]+\gamma(\gamma-1)(\gamma-2)\Psi=f \quad \text{(A2)}$$

where *a, b, c* and *G* are arbitrary constants, has a particular solution of the form (A-1) to (A-3), where *R* to *Z* are given by (34) to (36)

Corollary 3-2. If $f\in\mathcal{F}$ and $f_{-\gamma-1}\neq 0$, then the following of non-homogeneous third order differential equation

$$\Psi_3(z-a)(z-b)(z-c)+\Psi_2[3(\gamma+1)z^2-(a+b+c)(1+2\gamma)z+G]$$
$$+\Psi_1[3\gamma(\gamma+1)z-(a+b+c)\gamma^2]+\gamma(\gamma-1)(\gamma+1)\Psi=f \quad \text{(A3)}$$

where *a, b, c* and *G* are arbitrary constants, has a particular solution of the form (A-1) to (A-3), where *R* to *Z* are given by (38) to (40)

IV. Particular solution for the equation of the form $\Psi_3(z^2+az+b)(z-c)+\Psi_2(Dz^2+Ez+G)+\Psi_1(Hz+J)+\gamma(\gamma-1)(D-2\gamma-2)\Psi=f$ $a^2<4b$

Theorem 4. If $f\in\mathcal{F}$ and $f_{-\gamma-1}\neq 0$, then the following equation

$$\Psi_3(z^2+az+b)(z-c)+\Psi_2(Dz^2+Ez+G)+\Psi_1(Hz+J)+\gamma(\gamma-1)(D-2\gamma-2)\Psi=f \quad \text{(B)}$$

where *a, b, c, D, E, G, H* and *J* are given constants with the condition $a^2<4b$ and (B-7) has a particular solution of the form

$$\Psi=\left\{\left[(z^2+az+b)^{\frac{R}{2}}(z-c)^T\exp\left(\frac{S+\alpha R}{\beta}\tan^{-1}\frac{z-\alpha}{\beta}\right)\right]^{-1}\right.$$
$$\left.\cdot\left[(z^2+az+b)^{\frac{R}{2}-1}(z-c)^{T-1}\exp\left(\frac{S+\alpha R}{\beta}\tan^{-1}\frac{z-\alpha}{\beta}\right)f_{-\gamma}\right]_{-1}\right\}_{\gamma-2} \quad \text{(B-1)}$$

Where
$$\gamma=\frac{3J-AH}{3E-2AD} \quad \text{(B-2)}$$

$$A=a-c,\ B=b-ac,\ C=-bc \quad \text{(B-3)}$$

$$R=D-3\gamma-T \quad \text{(B-4)}$$

$$S=\frac{1}{c}(bT-G+B\gamma) \quad \text{(B-5)}$$

$$T=\frac{(D-3\gamma)c^2+(E-2A\gamma)c+G-B\gamma}{c^2+ac+b} \quad \text{(B-6)}$$

$$(3J - AH)^2 = (3E - 2AD)[(A - E)H - (3 - 2D)J] \qquad \text{(B-7)}$$

Proof: Suppose the non-homogeneous equation is

$$\Psi_3(z^3 + Az^2 + Bz + C) + \Psi_2[Dz^2 + Ez + G] + \Psi_1[Hz + J] + K\Psi = f \qquad (41)$$

where $z^3 + Az^2 + Bz + C = (z^2 + az + b)(z - c) \neq 0, \quad a^2 < 4b$

$$A = a - c,\ B = b - ac,\ C = -bc.$$

and $K = \gamma(\gamma - 1)(D - 2\gamma - 2)$

By proceeding in a manner similar to that in treating equation (7), we have

$$\left\{\omega_3 (z^2 + az + b)(z - c) + \omega_2[(D - 3\gamma)z^2 + (E - 2A\gamma)z + G - B\gamma]\right\}_\gamma = f. \qquad (42)$$

Let $U = \omega_2$, then $U_1 = \omega_3$ and (42) becomes

$$\left\{U_1(z^2 + az + b)(z - c) + U[(D - 3\gamma)z^2 + (E - 2A\gamma)z + G - B\gamma]\right\} = f_{-\gamma}. \qquad (43)$$

Hence $$U_1 + \frac{(D - 3\gamma)z^2 + (E - 2A\gamma)z + G - B\gamma}{(z^2 + az + b)(z - c)}U = \frac{f_{-\gamma}}{(z^2 + az + b)(z - c)}. \qquad (44)$$

Let $$p(z) = \frac{(D - 3\gamma)z^2 + (E - 2A\gamma)z + G - B\gamma}{(z^2 + az + b)(z - c)} = \frac{Rz + S}{(z - \alpha)^2 + \beta^2} + \frac{T}{z - c}, \qquad (45)$$

where $$\alpha = -\frac{a}{2}, \beta = \frac{\sqrt{4b - a^2}}{2} \qquad \alpha, \beta \in R \qquad (46)$$

and R, S, T are given by (B-4) to (B-6).

Then $$\int p(z)dz = \frac{1}{2}R\ln[(z - \alpha)^2 + \beta^2] + \frac{1}{\beta}(S + \alpha R)\tan^{-1}\frac{z - \alpha}{\beta} + \ln(z - c)^T, \qquad (47)$$

and $$U = \exp(-\int pdz)\left[\exp(\int pdz)\frac{f_{-\gamma}}{(z^2 + az + b)(z - c)}\right]_{-1}$$

$$= \left[(z^2 + az + b)^{\frac{R}{2}}(z - c)^T \exp(\frac{S + \alpha R}{\beta}\tan^{-1}\frac{z - \alpha}{\beta})\right]^{-1}$$

$$\cdot\left[(z^2 + az + b)^{\frac{R}{2}-1}(z - c)^{T-1}\exp(\frac{S + \alpha R}{\beta}\tan^{-1}\frac{z - \alpha}{\beta})f_{-\gamma}\right]_{-1} \qquad (48)$$

Hence we have a particular solution for (B) of the form

$$\Psi = \left\{\left[(z^2 + az + b)^{\frac{R}{2}}(z - c)^T \exp(\frac{S + \alpha R}{\beta}\tan^{-1}\frac{z - \alpha}{\beta})\right]^{-1}\right.$$

$$\left.\cdot\left[(z^2+az+b)^{\frac{R}{2}-1}(z-c)^{T-1}\exp\left(\frac{S+\alpha R}{\beta}\tan^{-1}\frac{z-\alpha}{\beta}\right)f_{-\gamma}\right]_{-1}\right\}_{\gamma-2}$$

As for γ, we see that the conditions (15) for $\omega_{\gamma+1}$ to be zero is the same, hence the results are identical with (A-4) and (A-11), namely

$$(3J-AH)^2=(3E-2AD)[(A-E)H-(3-2D)J]$$

and
$$\gamma=\frac{3J-AH}{3E-2AD}$$

This completes the proof.

An inspection of the proofs of Theorem 1 to Theorem 3 reveals that whenever the first term in (A), i.e., $(z-a)(z-b)(z-c)$ is unchanged, the forms of particular solutions are all identical. The only difference is that the parameters γ and R to Z depend on the coefficients a, b, c, D, E, G and H. The same argument can be applied. By means of Theorem 2, 3 and a little effort of calculation, we have the following theorems.

Theorem 5. If $f\in\mathcal{F}$ and $f_{-\gamma-1}\neq 0$, then equation (B) has a particular solution (B-1) provided
$$J=-A\gamma^2-(A-E)\gamma$$
where
$$\gamma=\frac{-(3-2D)\pm\sqrt{(3-2D)^2-12H}}{6}$$
$$A=a-c,\ B=b-ac,\ C=-bc$$

and R, S, T are given by (B-4), (B-5), (B-6).

Corollary 5-1. The equation (B) has a particular solution (B-1) provided
$$H=-3\gamma^2-(3-2D)\gamma$$
where
$$\gamma=\frac{-(A-E)\pm\sqrt{(A-E)^2-4AJ}}{2A},\quad A\neq 0,$$

and R, S, T are given by (B-4), (B-5), (B-6).

Theorem 6. If $f\in\mathcal{F}$ and $f_{-\gamma-1}\neq 0$, then the following of non-homogeneous third order differential equation
$$\Psi_3(z^2+az+b)(z-c)+\Psi_2[3(\gamma+\frac{1}{2})z^2+(a-c)(1+2\gamma)z+G]$$

$$+\Psi_1\gamma^2[3z+(a-c)]+\gamma(\gamma-1)(\gamma-\frac{1}{2})\Psi=f \tag{B1}$$

where *a, b, c* and *G* are arbitrary given constants with the condition $a^2<4b$, has a particular solution of the form (B-1)

where
$$R=\frac{3}{2}-T$$
$$S=\frac{1}{c}[bT-G+\gamma(b-ac)]$$
$$T=\frac{\frac{1}{2}c^2+ac+G-\gamma(b-ac)}{c^2+ac+b}$$

Corollary6-1. If $f\in\mathcal{F}$ and $f_{-\gamma-1}\neq 0$, then the following of non-homogeneous third order differential equation

$$\Psi_3(z^2+az+b)(z-c)+\Psi_2[3\gamma z^2+(a-c)(1+2\gamma)z+G]$$
$$+\Psi_1[3\gamma(\gamma-1)z+(a-c)\gamma^2]+\gamma(\gamma-1)(\gamma-2)\Psi=f \tag{B2}$$

where *a, b, c* and *G* are arbitrary given constants with the condition $a^2<4b$, has a particular solution of the form (B-1)

where
$$R=-T$$
$$S=\frac{1}{c}[bT-G+\gamma(b-ac)]$$
$$T=\frac{(a-c)c+G-\gamma(b-ac)}{c^2+ac+b}$$

Corollary 6-2. If $f\in\mathcal{F}$ and $f_{-\gamma-1}\neq 0$, then the following of non-homogeneous third order differential equation

$$\Psi_3(z^2+az+b)(z-c)+\Psi_2[3(\gamma+1)z^2+(a-c)(1+2\gamma)z+G]$$
$$+\Psi_1[3\gamma(\gamma+1)z+(a-c)\gamma^2]+\gamma(\gamma-1)(\gamma+1)\Psi=f \tag{B3}$$

where *a, b, c* and *G* are arbitrary given constants with the condition $a^2<4b$, has a particular solution of the form (B-1)

where
$$R=3-T$$
$$S=\frac{1}{c}[bT-G+\gamma(b-ac)]$$
$$T=\frac{2c^2+ac+G-\gamma(b-ac)}{c^2+ac+b}$$

V. Conclusion

Several second order linear differential equations have been solved by application of our method [7-10]. In trying to solve generalized third order linear differential equations in this paper, we see that at least two of the coefficients of the equations are restricted in order to have a particular solution. We expect that a further study of this method will enable us to solve equations with arbitrary parameters.

VI. References

[1] K. Nishimoto: Fractional Calculus, 1(1984), 4-6, 38-48, 154-165. 2(1987), 80-110. 3(1989), 118-143. 4(1991), 42-60, 74-80.

[2] K. Nishimoto: On Nishimoto's fractional calculus operator N^{ν} (on an action group), J. Frac. Calc. 4(1996), 1-11

[3] Shih-Tong Tu, Katsuyuki Nishimoto, S. J. Jaw and S. D. Lin: Applications of fractional calculus to ordinary and partial differential equations of the second order, Hiroshima Mathematical Journal, 23(1)(1993), 63-77.

[4] B. AL- SAQABI and S. L. Kalla: Solution of second and third order linear differential equations, Tamkang J. Math., 21(3)(1990), 191-200.

[5] Chung-Yi Lee: Particular solutions of Generalized linear second order differential equations by fractional calculus. Journal of Southeast University (English edition) 14(1998), 99~105.

[6] Chung-Yi Lee: Application of fractional calculus to linear third order (non-homogeneous and homogeneous) ordinary differential equation. Journal of Southeast University (English edition), 15(1999), 115~120.

[7] Chung-Yi Lee: Particular solutions of Riemann's differential equations by N-fractional calculus operator N^{α}, Journal of Shu-Teh, 24(1999).

[8] K. Nishimoto: Application of fractional calculus to differential equations of Fuchs type $\Psi_2 z^3 + \Psi_1(2(1+\alpha)z^2 + (\alpha+\beta+\gamma)z + \alpha\beta\gamma) + \Psi(\alpha(1+\alpha)z + \alpha(\alpha+\beta+\gamma)) = f$ J. Coll. Engng. Nihon Univ. B-28 (1987), 9-13.

[9] K. Nishimoto: An application of fractional calculus to differential equations of Fuchs type $\Psi_2(z^2-z) + \Psi_1(2Vz-V-1) + \Psi V(V-1) = f$. J. Coll. Engng.

Nihon Univ., B-27 (1986), 5-16.
[10] K. Nishimoto: Application of fractional calculus to a differential equation of Fuchs type $\Psi_2(z^2-1)+\Psi_1 2z-\Psi v(v+1)=f$. J. Coll. Engng. Nihon Univ. B-29 (1998), 9-10.

以分數微積分 N^{γ} 運算子求線性三階微分方程式特解的研究

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摘　要

分數微積分用於求解微分方程式已有不少著述。本文利用 K. Nishimoto 教授之 N^{γ} 運算子求解某類線性三階微分方程式，並得出十分簡捷之結果。

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